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ON CENTRES AND LINES OF MEAN POSITION.

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§ 1. INTRODUCTORY.

In this paper we shall define the centre of mean position of a number of points in a plane, with respect to a line of the plane, to be the centre of gravity of the points, supposing them to have weights inversely proportional to their distances from the line, points on one side of the line having positive and those on the other negative weights. Let us take a triangle of reference of which the line in question shall be one of the sides, and calling it the axis of z, let the axes of x and y be any two other straight lines. Calling the co-ordinates of the n points $x_1, y_1, z_1, \ldots x_n, y_n, z_n$, and those of the centre of mean position X, Y, Z, we have

$$X = \frac{\frac{x_1}{z_1} + \ldots + \frac{x_n}{z_n}}{\frac{1}{z_1} + \ldots + \frac{1}{z_n}},$$

$$Y = \frac{\frac{y_1}{z_1} + \ldots + \frac{y_n}{z_n}}{\frac{1}{z_1} + \ldots + \frac{1}{z_n}},$$

$$Z = \frac{\frac{z_1}{z_1} + \ldots + \frac{z_n}{z_n}}{\frac{1}{z_1} + \ldots + \frac{1}{z_n}};$$

or

$$\frac{X}{Z} = \frac{\frac{x_1}{z_1} + \ldots + \frac{x_n}{z_n}}{n},$$

$$\frac{Y}{Z} = \frac{\frac{y_1}{z_1} + \ldots + \frac{y_n}{z_n}}{n};$$

which latter expression we shall use. If the straight line of reference be the line at infinity, the centre of mean position is the centre of gravity of the points, supposing them all to have equal weights.

The same thing can be done for lines. We will take three points of reference $\alpha = 0$, $\beta = 0$, and $\gamma = 0$ in tangential co-ordinates, and will take the co-ordinates of any line to be the perpendiculars from the three points upon the line; taking the perpendicular from $\gamma = 0$ positive, and the perpendiculars from the others positive or negative according as they lie on the same side of the line with $\gamma = 0$ or on the opposite. We will define the co-ordinates of the mean line of position with respect to $\gamma = 0$ of any lines $\alpha_1, \beta_1, \gamma_1, \ldots \alpha_n, \beta_n, \gamma_n$, to be

$$\frac{a}{\Gamma} = \frac{\sum_1^n \frac{\alpha}{\gamma}}{n}, \qquad \frac{\beta}{\Gamma} = \frac{\sum_1^n \frac{\beta}{\gamma}}{n}.$$

§ 2.

This being premised, let

$$x^n f(u) + x^{n-1} z f_1(u) + x^{n-2} z^2 f_2(u) + \ldots = 0$$

be the equation of a plane curve of the nth degree in trilinear co-ordinates, having written $u \equiv y/x$. If we wish to get the tangents to the curve where the axis of z crosses it, we proceed as follows: Substitute

$$\frac{y}{x} = a + a' \frac{z}{x},$$

and determine a and a' so that the resulting equation in $\frac{x}{z}$ may have two infinite roots. The equation in x/z will be

$$\left[\frac{x}{z}\right]^n f(a) + \left[\frac{x}{z}\right]^{n-1} \Big[a' f'(a) + f_1(a)\Big]$$
$$+ \left[\frac{x}{z}\right]^{n-2} \left[\frac{a'^2}{2} f''(a) + a' f'_1(a) + f_2(a)\right] + \ldots = 0,$$

where we have written $f'(a) = df(a)/da$. In order that the equation in x/z may have two infinite roots, we must have

$$f(a) = 0 \quad \text{and} \quad a' f'(a) + f_1(a) = 0, \tag{1}$$

which will determine in general both a and a'. If it happen that the axis of z cut the curve in n points of inflexion, we shall also have

$$a'^2 f''(a) + 2a' f_1'(a) + 2 f_2(a) = 0$$

for all the sets of values of a and a'.

Let us now consider the points of contact of tangents from any point x', y', z' in the plane to the curve. They will be the intersections of the curve $U=0$ with the first polar of the point; namely,

$$x'\frac{\partial U}{\partial x}+y'\frac{\partial U}{\partial y}+z'\frac{\partial U}{\partial z}=0\,;$$

or we may write it

$$\left[\frac{x}{z}\right]^{n-1}F(u)+\left[\frac{x}{z}\right]^{n-2}F_1(u)+\ldots=0,$$

where

$$F(u)\equiv nx'f(u)+(y'-ux')f'(u)+z'f_1(u)$$

and

$$F_1(u)\equiv(n-1)\,x'f_1(u)+(y'-ux')\,f_1'(u)+2z'f_2(u),$$

where $u\equiv y/x$.

Now Liouville has shown that if we eliminate y between this equation and the equation of the curve we shall get

$$-\Sigma'\frac{x}{z}=\Sigma'\frac{a'F'(a)+F_1(a)}{F(a)}\,, \tag{2}$$

where the second Σ' pertains to all the sets of values of a and a' derived from the equations

$$f(a)=0 \quad\text{and}\quad a'f'(a)+f_1(a)=0.$$

Substituting for F and F_1 their value in (2), we have

$$-\Sigma'\frac{x}{z}=\Sigma'\frac{(y'-ax')\,(a'f''(a)+f_1'(a))+z'\,(a'f_1'(a)+2f_2(a))}{f'(a)\,(y'-ax')+z'f_1(a)}$$

$$=\Sigma'\frac{a'f''(a)+f_1'(a)}{f'(a)}+\Sigma'\frac{a'^2f''(a)+2a'f_1'(a)+2f_2(a)}{f'(a)(y'-ax')+z'f_1(a)}\,z'.$$

If the curve have n points of inflexion on the axis of z, we shall have

$$a'^2f''(a)+2a'f_1'(a)+2f_2(a)=0$$

for all the sets of values of a and a', for they are determined by the same equations (1) as before. In this case we see that $\Sigma'(x/z)$ is constant and independent of the point prime from which the tangents are drawn. We shall take

the axis of z to be the line to which the centre of mean position of the points of contact is referred, and therefore we shall have

$$\frac{X}{Z} = \frac{\Sigma \frac{x}{z}}{n(n-1)} \quad \text{and} \quad \frac{Y}{Z} = \frac{\Sigma \frac{y}{z}}{n(n-1)}.$$

We have therefore the theorem, that if a curve of the nth degree have n points of inflexion upon a right line, the mean centre of position with respect to that line, of the points of contact of tangents from any point in the plane to the curve, will be independent of the point.

By using line co-ordinates we would have obtained the theorem: If the tangents at n cuspidal points on a curve of the nth class meet in a point, the mean line of position with respect to this point of the tangents at the points where any line crosses the curve, will be independent of the line.

Every non-singular and nodal cubic has three points of inflexion upon a right line, and therefore we have as a corollary the theorem, that if tangents be drawn from any point in the plane to a non-singular or nodal cubic, the mean centre of position of the points of contact with reference to the line upon which three inflexions lie, is independent of the point. It may be interesting perhaps to see what this point is for the cubics in question. Every non-singular cubic can be put in the form

$$x^3 + y^3 + z^3 + 6mxyz = 0,$$

each of the axes of reference cutting the cubic in three points of inflexion. In this case we have

$$f(a) \equiv a^3 + 1, \qquad f_1(a) \equiv 6ma, \qquad f_2(a) \equiv 0.$$

a' is determined by the equation

$$3a^2 a' + 6ma = 0,$$

or

$$a' = -\frac{2m}{a}.$$

We have then

$$n(n-1)\frac{X}{Z} = -\Sigma \frac{a' f''(a) + f_1'(a)}{f'(a)}$$

$$= -\Sigma \frac{-\frac{12m}{a}a + 6m}{3a^2}$$

$$= 2m\Sigma \frac{1}{a^2} = 2m\Sigma a^2 = 0.$$

Likewise

$$\frac{Y}{Z} = 0.$$

We have therefore the theorem, that the common mean centre of position of the points of contact of tangents from any point in the plane to a non-singular cubic, with reference to a line upon which three inflexions lie, coincides with the point of intersection of the two lines upon which the other six inflexions lie. Also, that the line joining any point on this cubic with the centre of mean position of the points of contact of the four tangents (other than the tangent at the point itself) from it to the curve will pass through a fixed point. In the case of the nodal cubic the common mean centre is readily seen to be the node itself. For if we consider the points of contact of tangents from it to the curve, we see that all six points coincide with the node itself. From any point on a nodal cubic can be drawn two tangents to the curve other than the tangent at the point. Therefore the two tangents from a point on a nodal cubic to the curve, and the two lines joining the point with the node and with the point where the line joining the points of contact cuts the line upon which the three inflexions lie, form an harmonic pencil.

From the above follows the reciprocal theorem. Every curve of the 3rd class which possesses neither double tangents nor inflexions, has 9 cusps the tangents at which meet by threes in twelve points. If we take the point where any three meet as the point of reference, then the mean line of position with respect to it of the tangents at the points where any line crosses the curve coincides with the line joining the pair of points in which the other six cuspidal tangents meet.

§ 3.

We have found the following expression for the mean centre of position of the points of contact of tangents from the point x', y', z', to the curve with respect to the axis of z:

$$-n(n-1)\frac{X}{Z} = \Sigma\frac{a'f''(a)+f_1'(a)}{f'(a)} + \Sigma\frac{a'^2f''(a)+2a'f_1'(a)+2f_2(a)}{f'(a)(y'-ax')+z'f_1(a)}z'.$$

This expression will be constant for all points on the axis of z, since z' is a factor in the variable part. Therefore we have the theorem, that if tangents be drawn from any point on a given line to a curve of any degree, the mean centre of position of the points of contact with reference to that line will be independent of the point. This is an extension of the theorem of Chasles that "the centre of gravity of the points of contact of parallel tangents to any curve is independent of the direction of the tangents," since when the given line is at infinity the mean centre of position with regard to it coincides with the centre of gravity of the points supposing them to have equal weights, and the tangents from any point on it are parallel.

We will now find the mean centre of position of the poles of a line with respect to the line. We will choose the line for the axis of z. Let

$$U \equiv x^n f(u) + x^{n-1} z f_1(u) + \ldots = 0$$

be the equation of a curve of the nth degree, then will the poles of the axis of z with respect to the curve be given by the solution of the equations

$$\frac{\partial U}{\partial x} - \frac{u}{x}\frac{\partial U}{\partial u} = 0 \quad \text{and} \quad \frac{1}{x}\frac{\partial U}{\partial u} = 0,$$

which are the first polars of the points where the axes of x and y cross the axis of z.

These equations are equivalent to

$$x^{n-1}(nf - uf') + x^{n-2}z((n-1)f_1 - uf_1') + \ldots = 0$$

and

$$x^{n-1}f' + x^{n-2}zf_1' + \ldots = 0.$$

Writing $\quad F \equiv nf - uf', \qquad F_1 \equiv (n-1)f_1 - uf_1';$

we have

$$-\Sigma'\frac{x}{z} = \Sigma'\frac{a'F'(a) + F_1(a)}{F(a)},$$

where a and a' are determined by the equations

$$f'(a) = 0 \quad \text{and} \quad a'f''(a) + f_1'(a) = 0.$$

Substituting the values of F and F_1 we have

$$-\Sigma'\frac{x}{z} = \Sigma'\frac{a'[(n-1)f' - af''] + (n-1)f_1 - af_1'}{nf - af''}$$

$$= \frac{n-1}{n}\Sigma'\frac{f_1(a)}{f(a)};$$

or, since $\quad \dfrac{X}{Z} = \dfrac{\Sigma'\frac{x}{z}}{(n-1)^2}, \qquad -\dfrac{X}{Z} = \dfrac{\Sigma'\frac{f_1(a)}{f(a)}}{n(n-1)}.$

The mean centre of position of the points of contact of tangents from any point in the line with respect to the line might have been obtained in a different form by reversing the process of elimination. Thus the two equations being

$$x^n f + x^{n-1}zf_1 + \ldots = 0$$

and

$$x^{n-1}f' + x^{n-2}zf_1' + \ldots = 0,$$

we have

$$-\Sigma'\frac{x}{z}=\Sigma'\frac{a'f''(a)+f_1(a)}{f(a)},$$

where a and a' are determined by the equations

$$f'(a)=0 \quad \text{and} \quad a'f''(a)+f_1(a)=0,$$

or

$$-\Sigma'\frac{x}{z}=\Sigma'\frac{f_1(a)}{f(a)};$$

$$\therefore \quad -\frac{X}{Z}=\frac{1}{n(n-1)}\Sigma'\frac{f_1(a)}{f(a)};$$

and since a is determined by the same equation as before, we have the theorem that the mean centre of position of the points of contact of tangents to a curve from any point on a given line with respect to that line coincides with the mean centre of position with respect to the line of the poles of the line with respect to the curve. This point can readily be shown to be the tangential pole of the line.

In tangential co-ordinates any point has $(n-1)^2$ polar lines. We have as a reciprocal theorem then, that the mean line of position with respect to a point of the polar lines of the point with respect to the curve coincides with the mean line of position with respect to the point, of the tangents at the points where any line through the given point cuts the curve. This mean polar, reciprocally, is readily seen to be the trilinear polar line of the point.

§ 4.

These ideas may be readily extended to space. Let us define the mean centre of position of any number of points in space with respect to a plane, to be the centre of gravity of the points supposing them to have weights inversely proportional to their distances from the plane, points on one side having positive, and those on the other, negative weights. As before, calling the co-ordinates of the points $x_1, y_1, z_1, w_1, \ldots x_n, y_n, z_n, w_n$, and those of the centre of mean position X, Y, Z, W, we shall have

$$\frac{X}{W}=\frac{\frac{x_1}{w_1}+\ldots+\frac{x_n}{w_n}}{n},$$

$$\frac{Y}{W}=\frac{\frac{y_1}{w_1}+\ldots+\frac{y_n}{w_n}}{n},$$

$$\frac{Z}{W}=\frac{\frac{z_1}{w_1}+\ldots+\frac{z_n}{w_n}}{n}.$$

$$\text{Let } U \equiv \left[\frac{x}{w}\right]^{n} f(u, v) + \left[\frac{x}{w}\right]^{n-1} f_1(u, v) + \left[\frac{x}{w}\right]^{-} f_2(u, v) + \ldots = 0,$$

where
$$u \equiv \frac{y}{x} \quad \text{and} \quad v \equiv \frac{z}{x},$$

be the equation of a surface of the nth degree in homogeneous co-ordinates. The first polar of the intersection of the x, y, and z planes with respect to the surface is

$$\left[\frac{x}{w}\right]^{n-1} f_1(u, v) + 2\left[\frac{x}{w}\right]^{n-2} f_2(u, v) + \ldots = 0.$$

We get the tangents to the curve of intersection at the points where it crosses the w plane as follows:

Let

$$\frac{y}{x} = \alpha + \alpha' \frac{w}{x}, \qquad \frac{z}{x} = \beta + \beta' \frac{w}{x}$$

be the equations of any line. Substituting these values in the equation of the surfaces, we shall have

$$x^n f(\alpha, \beta) + w x^{n-1}\left[\alpha' \frac{\partial f}{\partial \alpha} + \beta' \frac{\partial f}{\partial \beta} + f_1\right] + w^2 x^{n-2}\left[\frac{\alpha' A + \beta' B + C}{2}\right] + \ldots = 0,$$

where

$$A \equiv \alpha' \frac{\partial^2 f}{\partial \alpha^2} + \beta' \frac{\partial^2 f}{\partial \alpha \partial \beta} + \frac{\partial f_1}{\partial \alpha},$$

$$B \equiv \alpha' \frac{\partial^2 f}{\partial \alpha \partial \beta} + \beta' \frac{\partial^2 f}{\partial \beta^2} + \frac{\partial f_1}{\partial \beta},$$

$$C \equiv \alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2,$$

and

$$x^{n-1} f_1(\alpha, \beta) + x^{n-2} w\left[\alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2\right] + \ldots = 0;$$

understanding any f written without anything following it to be expressed in terms of α and β.

In order that the line may touch the curve of intersection at the w plane, α, β, α', and β' are determined by the equations

$$f(\alpha, \beta) = 0, \qquad f_1(\alpha, \beta) = 0,$$

$$\alpha' \frac{\partial f}{\partial \alpha} + \beta' \frac{\partial f}{\partial \beta} + f_1 = 0, \quad \text{and} \quad \alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2 = 0.$$

But if the line touches the surface $U = 0$ in three points at every one of the points where the curve crosses the w plane, we shall also have

$$\alpha' A + \beta' B = 0$$

for all the sets of values of α, α', β, and β'.

This being premised, let x', y', z', w' be any point in space. The points of contact of tangent planes through the line joining x', y', z', w' with the intersection of the x, y, and z planes, will be given by the solution of the equations

$$U \equiv x^n f(u, v) + w x^{n-1} f_1(u, v) + w^2 x^{n-2} f_2(u, v) + \ldots = 0,$$

$$x^{n-1} f_1(u, v) + 2w x^{n-2} f_2(u, v) + \ldots = 0,$$

and

$$x^{n-1} \varphi(u, v) + w x^{n-2} \varphi_1(u, v) + \ldots = 0;$$

where $$\varphi(u, v) \equiv n x' f + (y' - u x') \frac{\partial f}{\partial u} + (z' - v x') \frac{\partial f}{\partial v} + w' f_1,$$

$$\varphi_1(u, v) \equiv (n - 1) x' f_1 + (y' - u x') \frac{\partial f_1}{\partial u} + (z' - v x') \frac{\partial f_1}{\partial v} + 2 w' f_2.$$

From these equations we have

$$-\Sigma' \frac{x}{w} = \Sigma' \frac{\alpha' \dfrac{\partial \varphi}{\partial \alpha} + \beta' \dfrac{\partial \varphi}{\partial \beta} + \varphi_1}{\varphi},$$

where the second Σ' extends to all the sets of values of α, α', β, and β' derived from the equations

$$f(\alpha, \beta) = 0, \qquad f_1(\alpha, \beta) = 0,$$

$$\alpha' \frac{\partial f}{\partial \alpha} + \beta' \frac{\partial f}{\partial \beta} = 0, \quad \text{and} \quad \alpha' \frac{\partial f_1}{\partial \alpha} + \beta' \frac{\partial f_1}{\partial \beta} + 2 f_2 = 0.$$

Substituting the values of φ and φ_1 we have

$$-\Sigma' \frac{x}{w} = \Sigma' \frac{(y' - \alpha x') A + (z' - \beta x') B}{(y' - \alpha x') \dfrac{\partial f}{\partial \alpha} + (z' - \beta x') \dfrac{\partial f}{\partial \beta}}$$

$$= \Sigma' \frac{A}{\dfrac{\partial f}{\partial \alpha}} + \Sigma' \frac{(\alpha' A + \beta' B)(z' - \beta x')}{\left[(y' - \alpha x') \dfrac{\partial f}{\partial \alpha} + (z' - \beta x') \dfrac{\partial f}{\partial \beta}\right] \beta'},$$

where A and B have the same values as given above. If the condition mentioned above is fulfilled, we shall have $\alpha' A + \beta' B = 0$ for all the sets of values of α, α', β, and β', since they are determined by exactly the same equations as

before. Therefore, if we define the co-ordinates of the mean centre by the equations

$$n(n-1)^2 \frac{X}{W} = \Sigma' \frac{x}{w},$$

and similarly for Y and Z, we shall have the following theorem:

If the first polar of a point with respect to a surface cut the surface at $n(n-1)$ points in a plane along the asymptotic lines, the mean centre of position with reference to this plane of the points of contact of tangent planes to the surface through any line through this point, will be independent of the line.

§ 5.

The extension of the theorem concerning parallel tangent planes to a surface may be obtained as follows: Let

$$x^n f(u, v) + x^{n-1} w f_1(u, v) + x^{n-2} w^2 f_2(u, v) + \ldots = 0$$

be the equation of a surface of the nth degree. The first polar of the intersection of the x, z, and w planes will be

$$x^{n-1} \frac{\partial f}{\partial u} + x^{n-2} w \frac{\partial f_1}{\partial u} + \ldots = 0.$$

Let us take any other point in the w plane x', y', z', 0. The co-ordinates of the points of contact of tangent planes through the line joining these two points will be given by the solution of the above equations and the first polar of x', y', z', 0 with respect to the surface; namely,

$$x^{n-1} \varphi(u, v) + x^{n-2} w \varphi_1(u, v) + \ldots = 0,$$

where

$$\varphi \equiv n x' f + (y' - u x') \frac{\partial f}{\partial u} + (z' - v x') \frac{\partial f}{\partial v},$$

and

$$\varphi_1 \equiv (n-1) x' f_1 + (y' - u x') \frac{\partial f_1}{\partial u} + (z' - v x') \frac{\partial f_1}{\partial v}.$$

From these we get

$$-\Sigma' \frac{x}{w} = \Sigma' \frac{\alpha' \frac{\partial \varphi}{\partial \alpha} + \beta' \frac{\partial \varphi}{\partial \beta} + \varphi_1}{\varphi},$$

where α, β, α', and β' are given by the equations

$$f(\alpha, \beta) = 0, \qquad \alpha' \frac{\partial f}{\partial \alpha} + \beta' \frac{\partial f}{\partial \beta} + f_1 = 0,$$

$$\frac{\partial f}{\partial \alpha} = 0, \quad \text{and} \quad \alpha' \frac{\partial^2 f}{\partial \alpha^2} + \beta' \frac{\partial^2 f}{\partial \alpha \partial \beta} + \frac{\partial f_1}{\partial \alpha} = 0.$$

Substituting the values of φ and φ_1, we shall find the multipliers of x' and $y' - ax'$ in both numerator and denominator of $\Sigma\,(x/w)$ vanish, and we have

$$-\Sigma\frac{x}{w}=\Sigma\,\frac{\left[\alpha'\dfrac{\partial^2 f}{\partial\alpha\partial\beta}+\beta'\dfrac{\partial^2 f}{\partial\beta^2}+\dfrac{\partial f_1}{\partial\beta}\right](z'-\beta x')}{(z'-\beta x')\dfrac{\partial f}{\partial\beta}},$$

which is independent of the point x', y', z', 0.

Therefore we have the theorem, that if tangent planes be drawn through any line in a given plane passing through a given point in the plane, the centre of mean position of the points of contact with reference to the plane, will be independent of the line. If the plane be the plane at infinity this becomes the well-known theorem, that the mean centre of the points of contact of parallel tangent planes to a surface is independent of the direction of the planes.

By reversing the process of elimination we get the co-ordinates of the mean centre of position of any line passing through the intersection of the x, z, and w planes, and lying in the w plane in a different form, by the solution of

$$x^n f + x^{n-1} w f_1 + \ldots = 0\,,$$

$$x^{n-1}\frac{\partial f}{\partial u} + x^{n-2} w\,\frac{\partial f_1}{\partial u} + \ldots = 0\,,$$

and

$$x^{n-1}\frac{\partial f}{\partial v} + x^{n-2} w\,\frac{\partial f_1}{\partial v} + \ldots = 0\,.$$

From these we obtain

$$-\Sigma\frac{x}{w}=\Sigma\,\frac{\alpha'\dfrac{\partial f}{\partial\alpha}+\beta'\dfrac{\partial f}{\partial\beta}+f_1}{f},$$

where α, β, α', and β' are determined by the equations

$$\frac{\partial f}{\partial\alpha}=0\,,\qquad \frac{\partial f}{\partial\beta}=0\,;$$

and

$$\alpha'\frac{\partial^2 f}{\partial\alpha^2}+\beta'\frac{\partial^2 f}{\partial\alpha\partial\beta}+\frac{\partial f_1}{\partial\alpha}=0\,,$$

$$\alpha'\frac{\partial^2 f}{\partial\alpha\partial\beta}+\beta'\frac{\partial^2 f}{\partial\beta^2}+\frac{\partial f_1}{\partial\beta}=0\,;$$

$$\therefore\quad -\frac{X}{W}=\frac{1}{n\,(n-1)^2}\,\Sigma\frac{f_1}{f}\,,\quad\text{since}\quad \frac{\partial f}{\partial\alpha}=\frac{\partial f}{\partial\beta}=0\,.$$

We will now find the mean centre of position with respect to the plane of the poles of the plane with respect to the surface. The poles will be given by the solution of

$$x^{n-1}\frac{\partial f}{\partial u} + x^{n-2}w\frac{\partial f_1}{\partial u} + \ldots = 0\,,$$

$$x^{n-1}\frac{\partial f}{\partial v} + x^{n-2}w\frac{\partial f_1}{\partial v} + \ldots = 0\,,$$

and

$$x^{n-1}F + x^{n-2}wF_1 + \ldots = 0\,;$$

where

$$F \equiv nf - u\frac{\partial f}{\partial u} - v\frac{\partial f}{\partial v}\,,$$

and

$$F_1 \equiv (n-1)f_1 - u\frac{\partial f_1}{\partial u} - v\frac{\partial f_1}{\partial v}\,.$$

We have, then,

$$-\Sigma'\frac{x}{w} = \Sigma'\frac{\alpha'\dfrac{\partial F}{\partial \alpha} + \beta'\dfrac{\partial F}{\partial \beta} + F_1}{F}\,,$$

where α, α', β, and β' are determined by exactly the same equations as before. Substituting the values of F and F_1, we find

$$-\Sigma'\frac{x}{w} = \frac{n-1}{n}\Sigma'\frac{f_1}{f}\,;$$

$$\therefore\; -\frac{X}{W} = \frac{\dfrac{n-1}{n}\Sigma'\dfrac{f_1}{f}}{(n-1)^3} = \frac{1}{n(n-1)^2}\Sigma'\frac{f_1}{f}\,.$$

Therefore we have the theorem that the centre of mean position with respect to a plane of the points of contact of tangent planes to a surface through any line in the plane is independent of the line, and coincides with the mean centre of position with respect to the plane of the poles of the plane with respect to the surface. An obvious consequence is that the mean centre of the points of contact of parallel tangent planes coincides with the mean centre of the poles of the plane at infinity. As in the plane, this point is readily seen to be the tangential pole of the plane.

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